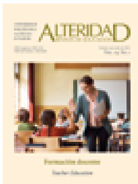
ALTERIDAD
REVISTA DE EDUCACIÓN
p-ISSN:1390-325X / e-ISSN:1390-8642
http://alteridad.ups.edu.ec

January-July 2024
Vol. 19, No. 1, 45-56
https://doi.org/10.17163/alt.v19n1.2024.04



Guideline to reflect on the education functions and improve their teaching

Pauta para reflexionar sobre la enseñanza de las funciones y mejorar su docencia

Neus Inglada Rodríguez is a professor at Universidad de Barcelona, Spain (ninglada@ub.edu) (https://orcid.org/0000-0002-5741-7531)

Dra. Adriana Breda is a professor at Universidad de Barcelona, Spain (adriana.breda@ub.edu) (https://orcid.org/0000-0002-7764-0511)

Dra. Gemma Sala-Sebastià is a professor at Universidad de Barcelona, Spain (gsala@ub.edu) (https://orcid.org/0000-0001-9830-312X)

Received on: 2023-09-29 / **Revised on:** 2023-12-09 / **Accepted on:** 2023-12-13 / **Published on:** 2024-01-01

Abstract

Teachers should have the competence to reflect on their own practice. The Ontosemiotic Approach offers as a tool for this purpose the "Didactic Suitability Criteria". The objectives of the research are to refine the indicators of epistemic suitability (ES) and to deepen the epistemic dimension of the meta-didactic-mathematical knowledge of secondary school mathematics teachers in initial training. To this end, research on the object function in secondary education has been examined. In a mixed way, we have analysed, in 119 "Master's Dissertation (MD) in Secondary Education Mathematics Teachers in service Training (Catalonia, Spain)" on functions, the reflection that future teachers make on their teaching, from the ES. Based on these analyses, the adequacy of the epistemic suitability for analysing the mathematical object function is designed. Finally, the MDs are reanalysed with this new guideline and it is found that their reflections present important weaknesses that could influence the quality of their instructional processes. It is concluded that if mathematics teachers were provided with a specialised guideline that makes it easier for them to consider all the meanings, representations and processes involved in the complexity of the functions, as well as the mathematical practices in which these emerge, it would improve the quality of the design, implementation, and reflection on their instructional processes.

Keywords: mathematics education, secondary education teachers in service training, initial teacher training, analysis of functions, epistemology, Didactic Suitability Criteria.

Resumen

El profesorado debe tener la competencia de reflexionar sobre su propia práctica. El enfoque ontosemiótico ofrece como herramienta para este propósito los "Criterios de Idoneidad Didáctica". Los objetivos de esta investigación son refinar los indicadores de la Idoneidad Epistémica (IE) y profundizar en la dimensión epistémica del conocimiento meta didáctico-matemático de los profesores de matemáticas de secundaria en formación inicial. Para ello se han examinado las investigaciones sobre el objeto función en la educación secundaria. De forma mixta, se ha analizado, en 119 "Trabajos Finales del Máster (MD) de Formación del Profesorado de Matemáticas de Secundaria (Catalunya, España)" sobre funciones, la reflexión que hacen los futuros profesores sobre su docencia, sobre la ES. A partir de estos estudios se realiza el diseño de la adecuación de la ES para analizar el objeto matemático función. Finalmente se reanalizan los MD con esta nueva pauta y se constata que sus reflexiones presentan importantes carencias que podrían influir en la calidad de sus procesos de instrucción. Se concluye que, si se dotara al profesorado de una pauta especializada que les facilite considerar todos los significados, representaciones y procesos involucrados en la complejidad de las funciones, así como las prácticas matemáticas en las que estos emergen, mejoraría la calidad del diseño, implementación y reflexión sobre sus procesos de instrucción.

Palabras clave: educación matemática, formación de docentes de secundaria, formación preparatoria de docentes, análisis funcional, epistemología, Criterios de Idoneidad Didáctica.

Suggested citation (APA): Inglada Rodríguez, N., Breda, A. & Sala-Sebastià, G. (2024). Guideline to reflect on the education functions and improve their teaching. *Alteridad, 19*(1), 45-56. https://doi.org/10.17163/alt.v19n1.2024.04

1. Introduction

Several authors explain through different theoretical models what teachers must do in their profession (Shulman, 1987; Mishra & Koehler, 2006). Amaya de Armas et al. (2016) point out the importance of identifying the knowledge of a mathematics teacher. For this purpose, there is the model "Knowledge and Competence Didactic-Mathematics of the Mathematics Professor" (DMKC) of Juan Godino et al. (2017), which is a refinement of the theoretical model of Deborah Ball et al. (2008).

Regarding the didactic dimension, mathematics teachers must have the competence to reflect on the processes of mathematical instruction carried out, because it is a fundamental strategy for professional growth and enrichment of teaching. In the research about the importance of the reflection of teachers on their teaching practice are the action research of Elliot et al. (1993), Schön's reflective practice (1983) and the study of lessons of Hart et al. (2011). The "Criteria of Didactic Suitability" (CID) offered by the framework Ontosemiotic Approach to Mathematical Knowledge and Instruction (OSA) of Godino et al. (2007), is an instrument designed to order and structure the reflection of teachers articulating different criteria (epistemic, cognitive, interactional, mediational, affective, and ecological) being Didactic-Mathematical Knowledge (DMK) one of the elements proposed by the OSA, called meta didactic-mathematical knowledge (Breda et al., 2017).

It is important to note that CID has been applied in different teacher training processes in various countries, obtaining satisfactory results in terms of the development of teacher reflection to increase the teaching quality: Ecuador and Spain (Font et al., 2023), Chile (Seckel & Font, 2020), Costa Rica (Morales-López & Font, 2019), Panama (Morales-Maure, 2019), Peru (Garcés-Córdova & Font, 2022). The "Interuniversity Master of Secondary Education Teachers of Mathematics in Catalonia" considers the criterion that future teachers must perform teaching practices in educational centers during their initial training, and that it is necessary to reflect on them to obtain the great complexity of teaching and learning processes.

To achieve this objective, the students of the Master's course take a subject called Final Master's Work (MD) in which they analyze the Didactic Unit (DU) developed and implemented by themselves in the pre-professional internship. To perform this analysis, students use DSCs that have studied in another subject. Future teachers, based on analysis, redesign their DU by improving it.

As stated by Font (2011), one of the most important topics in secondary mathematics education is that of functions. They are nuclear because they are present in many modeling processes, and because of their epistemic richness and complexity.

In this paper we analyze the functions of Compulsory Secondary Education (ESO, because of the Spanish acronym) in the MD, and the reflection that teachers in initial training make when they analyze epistemically the DU they have designed.

1.1 Mathematical-didactic knowledge and competency model (DMKC)

In order to improve the training of mathematics teachers, Pino-Fan et al. (2015) propose a DMK model that explains and determines the knowledge of a teacher considering three dimensions: mathematics, didactics and didactic-mathematics goal. In this research we will focus on the third dimension.

Different theoretical constructs have been proposed in OSA to develop this meta-didactic-mathematical dimension, particularly for evaluating instruction processes in mathematics, being its essential tool the notion of didactic suitability. It is said that a teaching and learning process has a certain level of Didactic Suitability if it has certain elements that make it possible to assess it as suitable, in the sense of appropriate or optimal, for students to transform the institutional meanings intended or implemented by the teacher (teaching) into personal meanings (learning), considering the circumstances and means (environment) (Godino et al., 2006a; Godino et al., 2006b). Teaching suitability is defined from the following dimensions or DSC: ES, evaluates the quality of the mathematics taught; Cognitive suitability, evaluates the previous knowledge of the students and if the students have learned; Interactional suitability, evaluates whether interactions between teacher-learners and between learners contribute to the learning of mathematics; Mediational suitability, evaluates the management of time and the suitability of materials and other resources used; Affective suitability, evaluates the degree of motivation and interest of students during teaching and learning; Ecological suitability, evaluates the adaptation from the teaching and learning process to the curriculum, to the school's ideology, to the socioeconomic context and to the future (Font et al., 2010). The analysis we present in this paper focuses on ES.

The ES studies the representativeness of the different meanings of mathematical objects present in the instruction process. For example, in the case of teaching the functions of 4th of ESO, the aim is to reduce teaching to the operational aspect and its algebraic representation (low suitability) or to work different meanings of function, such as correspondence, relationship between variables, relationship between magnitudes, and their different representations, verbal, algebraic, tabular, graphical and iconic (high suitability).

Breda et al. (2017) establish a structure of components and indicators that guide and organize the analysis and evaluation of the educational suitability of the study processes of any educational stage. It is important to bear in mind that the components and also the indicators of DSCs have been set considering the principles, trends and results of research in Mathematical Education (Breda et al., 2018). Table 1 presents the components and indicators of ES.

Table 1. *ES components and indicators*

Components	Indicators
Errors	There are no practices considered to be mathematically incorrect. Good practices (without errors) are observed from the mathematical point of view.
Ambiguities	Unambiguous practices are observed... There are no ambiguities that can lead to confusion for students: clear and correctly stated definitions and procedures, adapted to the educational level to which they are directed; adequacy of explanations, checks, demonstrations to the educational level to which they are directed, controlled use of metaphors, etc
Richness of processes	The sequence of tasks includes the performance of relevant processes in mathematical activity (modeling, argumentation, problem solving, connections, etc.).
Representativeness	Partial meanings (definitions, properties, procedures, etc.) are a representative sample of the complexity of the mathematical notion (indicated in the program) Partial meanings (definitions, properties, procedures, etc.) are a representative sample of the complexity of the mathematical notion. For one or more partial meanings, a representative sample of problems. For one or more partial meanings, use of different ways of expression (verbal, graphic, symbolic...), treatments and conversions between them.

Note. Breda et al. (2017, p.1093).

1.2 Research on the notion of function in the framework of OSA

Several studies have been conducted on the concept of function in the theoretical framework of OSA (Amaya de Armas et al., 2016; Flores & Font, 2017; Parra-Urrea & Pino-Fan, 2017; Pino-Fan & Parra-Urrea, 2021; Ramos & Font, 2008; Sánchez et al., 2021). Our research is based on previous works to deepen on the epistemic aspect of the notion of function and the processes involved in its teaching and learning. We collect the lists proposed in these investigations, classifications and characterizations of the processes related with the components of the ES to complete the initiative of Pino-Fan & Parra-Urrea (2021), by designing a tool that adapts the DSCs to analyze, evaluate and improve the instruction processes of functions. This tool will enable the research of teachers' meta didactic-mathematical knowledge.

2. Methodology

This research is mixed, since quantitative, descriptive and qualitative methods are used. Using the strengths of both approaches increases the quality of research (Leite et al., 2021). The quantitative methodology is applied to the selection and quantification of MDs that have developed their DU on ESO functions. However, the qualitative approach centered on the reflection of teachers in initial training predominates in this work. Thus, inductive categories of types of errors, ambiguities, richness of processes and representativeness of the complexity of the object and function worked in the ESO emerge from the analysis of the reflection that future teachers have included in their MD. The study, the comparison, and the generalization of these new categories have allowed us to design a specific tool for the planning, analysis and evaluation of the instruction processes of functions in the ESO.

2.1 Context and participants

The data refer to 119 MD on ESO functions of students coursing the "Interuniversity Training Master of the Mathematics Teacher of Secondary in Catalonia" from 2011-2012 to 2020-2021 academic year. The students of the master's degree carry out two practice phases in secondary schools. The aim of the first is for teachers in initial training to familiarize themselves with the school, the students and start working with the supervisor of the center in the DU that they must design. In the second phase, future teachers implement the DU they have prepared. Then, in the MD, they apply the DSCs to analyze the degree of Didactic Suitability of their own teaching practice and redesign the DU to raise the level of Didactic Suitability. When assessing ES, they reflect on errors, ambiguities, richness of processes and representativeness of the complexity of functions.

2.2 Design of the ES Refining Indicators for Functions (RIEF) tool

To design the RIEF we have adapted the steps of the thematic analysis prepared by Braun and Clarke (2006) structured in six phases. In the first step, a bibliographic study was made and the proposed indicators in Godino et al. (2006a), Pino-Fan & Parra-Urrea (2021) and Sánchez et al. (2021) were considered to make a first analysis of the MD; also, a list of indicators present in the reflections and proposals for improvement of teachers in initial training was elaborated from a triangulation of experts of the DSC tool. In the second step, from the two lists above, we have made a single listing. In a third step, we have classified the indicators by components of the ES criterion (errors, ambiguities, richness of processes and representativeness of the complexity of mathematical objects) and assigned an initial code according to the component to which it belongs (Ei), (Ai), (Pi) and (ROMi). In the fourth step, we have reviewed the indicators within each component. Some indicators were not in line with the assigned component. Some have been eliminated and others have been given a new category (appropriate didactic option (Oi), meanings (Mi), representations and conversions (RCi) and problem situations (Ti)). As seen, the category corresponding to the criterion representativeness of the complexity of mathematical objects has been replaced by three new categories: meanings, representations and conversions and problem situations. In the fifth step, we have worked on defining each of the indicators so that it is clear and operational. We have also reviewed the consistency within each category and globally of the entire tool. Finally, in the sixth step, we have structured the categories of indicators as a specialized guideline to reflect on the teaching of functions in secondary schools.

2.3 Analysis of MD using RIEF

In the first phase, a quantitative analysis of descriptive statistics is carried out. In particular, the absolute and relative frequency of MD is calculated, in which each of the RIEF indicators has been identified. It is distinguished whether it is present in the analysis of the planning and implementation of the DU or in the redesign. In the second phase, a qualitative analysis is performed from a triangulation of experts, which, from the evidence present in the MD, allows us to characterize the meta-didactic mathematical knowledge of its authors. We study how they help them to reflect on the indicators of the ES criterion and how the RIEF guideline facilitates a more guided, and therefore deeper analysis that would make more explicit the weaknesses and achievements of the instructional processes.

3 Results

In this research, three different levels of analysis have been performed. Thus, the results obtained in each of them are different. RIEF has been developed first, then MDs are analyzed in a quantitative way and, finally, the MDs are analyzed qualitatively.

3.1 RIEF indicators

We have obtained the following adaptation of the ES in order to facilitate the analysis of the instruction processes in the secondary compulsory about functions.

3.1.1 Errors

(E1) "The error of using continuous curves for discrete functions is avoided" (Pino-Fan & Parra-Urrea, 2021, p.50). (E2) Definition error. (E3) Rendering error. (E4) Resolution or procedure error. (E5) Error in the proposition of a problem. (E6) Argumentation error.

3.1.2 Ambiguities

(A1) Metaphors are used consciously. (A2) Using notation to represent the function and image of a value in the table without specifying the two meanings. (A3) Dynamic function language. (A4) Language inaccuracy. (A5) Using notation to represent a point and a range without specifying the two meanings.

3.1.3 Appropriate didactic option

(O1) "To work with functions is not limited to the use of algebraic representations to avoid them to be perceived only as formulas and regularities" (Pino-Fan & Parra-Urrea, 2021, p. 50). (O2) The "belief that a change in the independent variable necessarily implies a change in the dependent variable" is avoided (Pino-Fan & Parra-Urrea, 2021, p. 50). (O3) "Functional relationships that are not graphable are presented to avoid the belief that every function supports a graphical representation" (Pino-Fan & Parra-Urrea, 2021, p. 50). (O4) "Functional relationships that do not have an algebraic expression associated with them are presented to avoid the belief that every function supports an algebraic representation" (Pino-Fan & Parra-Urrea, 2021, p. 50). (O5) "Functions are presented with explicit domains to avoid the belief that every function has a domain and a natural or real codomain" (Pino-Fan & Parra-Urrea, 2021, p. 50). (O6) 'Irregular' graphs are presented to avoid the belief that any graphically represented function has 'good behavior' (Pino-Fan & Parra-Urrea, 2021, p. 50). (O7) "Definitions and procedures consider arbitrariness and univalence as key features of the notion of function" (Pino-Fan & Parra-Urrea, 2021, p. 50). (O8) "The notions of domain and codomain are presented as inherent elements to the definition of function" (Pino-Fan & Parra-Urrea, 2021, p.50). (O9) "Fundamental statements and procedures relating to the notion of function appropriate to the educational level are presented" (Pino-Fan & Parra-Urrea, 2021, p. 50). (O10) When introducing the Cartesian reference system, account is taken to any confusion that may occur (O11). Students lack prior knowledge that hinders learning.

3.1.4. Richness of processes

(P1) Problem statements are read and interpreted correctly. (P2) Conjectures and propositions are stated. (P3) Argumentation: conjectures and procedures are justified. (P4) Definitions and procedures are institutionalized. (P5) Variables and quantities are identified. (P6) It is identified whether a relationship is functional and if so, the type. (P7) Algorithms, routines or calculations are applied. (P8) Generalization and abstraction processes are carried out.

3.1.5. Meanings

(M1) "Function as correspondence" (Pino-Fan & Parra-Urrea, 2021, p. 47). (M2) The function as a relationship between variables. (M3) "Function as a ratio between magnitudes" (Pino-Fan & Parra-Urrea, 2021, p. 47). (M4) "Function as arbitrary correspondence" (Pino-Fan & Parra-Urrea, 2021, p. 47). (M5) "The function from the theory of sets" (Pino-Fan & Parra-Urrea, 2021, p. 47).

3.1.6. Representations and conversions

Representation is mobilized: (R1) verbal. (R2) algebraic. (R3) tabular. (R4) graphical. (R5) The type is not specified.

Conversions between verbal (R6) and algebraic representation are promoted. (R7) verbal and tabular. (R8) verbal and graphical. (R9) algebraic and tabular. (R10) algebraic and graphical. (R11) tabular and graphical. (R12) types are not specified.

3.1.7. Problem situations

The proposed problems (T1) activate the different meanings of function. (T2) mobilizes the different function representations and their conversions. (T3) 'in purely mathematical contexts to reinforce learning about functions' (Pino-Fan & Parra-Urrea, 2021, p. 50). (T4) where intramathematical connections are worked. (T5) 'contextualized from everyday life or other sciences' (Pino-Fan & Parra-Urrea, 2021, p. 50). (T6) of modeling. (T7) involving the different types of functions worked.

It is important to mention that the RIEF contains an appropriate *didactic option* category whose indicators do not correspond to ES but to the cognitive, but they are identified as errors or as ambiguities in the ES analysis of MDs; hence, these need to be taken into account in order to correct this trend. As indicated, three categories have emerged from the representativeness component of the complexity of mathematical objects (meanings, representations and conversions and situations) but, on the other hand, no propositions, procedures or arguments have emerged, elements that constitute epistemic configurations along with the three previous ones.

3.2 Results obtained from the quantitative analysis

Reflections on their own practice by authors of MD have been analyzed to identify which of the above indicators are used to propose improvements in their DU.

Table 2 shows the data collected in the analysis of the 119 MD participants in relation to the RIEF components and indicators identified in their reflections. We have counted the number of MD, where we reflect on each of the RIEF indicators (I) and in the aspects they are presented (design and implementation (D) and / or proposal for improvement (M) of the DU). In relation to the MD analyzed, X indicates that no reflection on the component has been found; F refers to the author of the MD, stating that it does not detect errors or ambiguities; V indicates that the will not to make errors or introduce ambiguities or enhance processes is made explicit. The data in column F indicate, for each category of RIEF indicators, the number of MDs that do not show any reflection on that category. The fourth and subsequent columns show the number of MDs where each RIEF indicator has been identified (see details of indicators in section 3.1).

Table 2. *Number of MDs reflecting on each RIEF indicator*

	I	**X**	**F**	**E1**	**E2**	**E3**	**E4**	**E5**	**E6**	**V**			
Errors	D	20	39	12	22	21	17	23	6	---			
	M	99	---	5	2	4	2	5	1	10			
	I	X	F	A1	A2	A3	A4	A5	V				
Ambiguities	D	56	3	30	5	32	27	3	---				
	M	88	---	16	1	5	3	1	12				
	I	X	O1	O2	O3	O4	O5	O6	O7	O8	O9	O10	O11
Didactic options	D	71	1	0	0	0	5	3	2	5	24	6	3
	M	110	0	0	0	0	1	0	0	2	2	3	2
	I	X	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	V		
Richness of processes	D	11	63	99	83	38	17	16	39	15	---		
	M	57	14	51	26	6	4	5	3	4	1		

I	X	F	E1	E2	E3	E4	E5	E6	V					
Meanings	I	X	S1	S2	S3	S4	S5							
	D	45	19	42	51	13	41							
	M	101	2	6	4	2	4							
Representations	I	X	R1	R2	R3	R4	R5	R6	R7	R8	R9	R10	R11	R12
	D	13	63	97	97	99	7	43	45	43	48	49	53	27
	M	44	5	10	4	6	6	8	8	8	9	9	10	6
Problem Situations	I	X	T1	T2	T3	T4	T5	T6	T7					
	D	9	14	10	17	68	63	50	16					
		58	13	16	9	26	27	20	12					

From table 2 it is inferred that many of the RIEF indicators are not considered in the analyzes that have been made by future teachers about their implementations. This is more evident in the redesigns they have proposed.

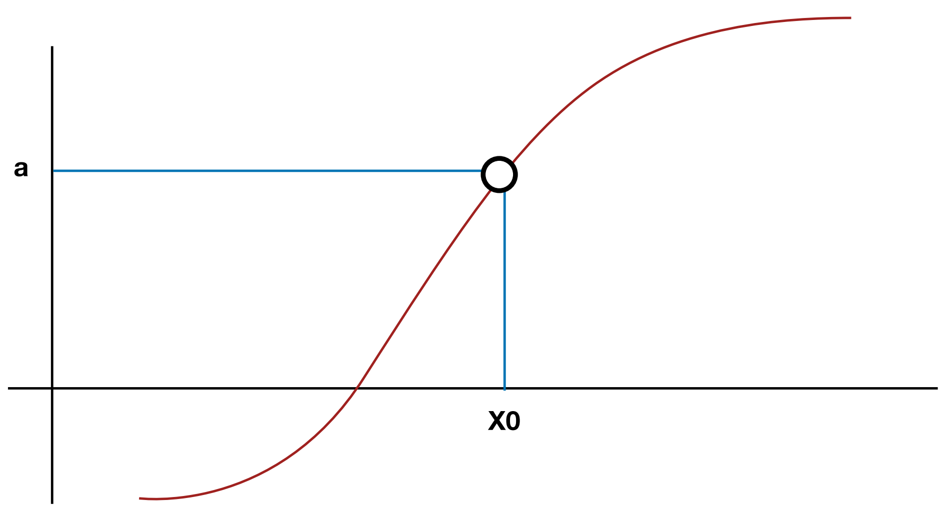
3.3 Results obtained from the qualitative analysis

The analysis of the reflection made by students of the master's degree in MD on ESO functions shows that the types of *errors* on which they reflect are mainly based on problem proposition (19%), definitions (19%) and representations (18%).

The types of *ambiguity* that are most reflected upon are the use of dynamic function language (27%) and the use of metaphors in a conscious way (25%). However, only 4% of future teachers reflect on the use of notation to represent the function and image of a value without specifying the two meanings. MD file 41937 describes the following ambiguity detected by the author in the implementation of DU.

> There was ambiguity in explaining the discontinuous functions, which we defined as those functions whose graph cannot be drawn without lifting the pen from the paper. This definition caused confusion when we drew a discontinuous function in which the point of discontinuity was identified by a circle [...]. Some students said that at no time did the pencil rise to continue drawing the whole function, as they went around the circle, and therefore it was not a discontinuous function.

Figure 1. *Discontinuous function?*



Note. MD file 41937 (Authors' translation).

The most appropriate *didactic option* in the reflection of MD refers to the adequacy to educational level (20%). The most used process is the enunciation of conjectures and propositions (83%), followed by the justification of conjectures and procedures (70%). Most future teachers recognize a lack of this type of process in the design and implementation of their DUs and 52% introduce them in their proposals for improvement. The author of MD file 21708 justifies the presence of these processes. This is how the argument is evidenced in the DU:

> The activity displayed the graph of a function on a few coordinate axes. The goal was to choose which sports (from a list) could be represented with that graph. Students were forced to argue and support their answers (MD file 21708, authors' translation)

The most common *meanings* in the reflections of the MD are the function as relation between magnitudes (43 %), relation between variables (35 %) and the theory of sets (35 %), although only 24 % of the authors who reflect on the meanings when analyzing their DU do so in their proposals for improvement. 89 % of the MD reflect on the representations and conversions component. It is in this component that there is more evidence of a more detailed analysis. Here is the author's reflection on MD file 11402:

Figure 2. *Conversions between representations*

A From	Table of values	Verbally	Graphically	Symbolic Expression
Table of values		NO	YES	NO
Verbally	YES		YES	YES
Graphically	YES	NO		NO
Symbolic Expression	NO	YES	YES	

Note. MD file 11402 (translation by the authors).

> As seen in Figure 2, we work on almost all the ways of representation and in all directions. The way we worked the most was to go from the verbal form to the other three forms of representation. And the one that we used the least was the symbolic expression and its conversion to the others, since the students of 1st year of ESO had not studied algebraic expressions. We can say that we work in 6/11 of the possible directions. (MD file 11402, authors' translation)

The *problem situations* component is the most present. 92% of the works contain reflections on some of its indicators. The most studied is the presentation of problems where intramathematical connections are worked (57 %) and it is followed by the presentation of contextualized problems (53 %).

It is very common that the analysis of the mathematical processes worked is limited to indicate which of the described in the following figure 3 have been present in the design and implementation of the DU. Some authors justify citing some activities as an example, but without more concreteness or deepening. The author of MD file 21913 makes a good analysis from the guideline contained in Figure 3, but since this guideline does not consider the mathematical objects being worked, it is generalist, unrefined and is not enough for teachers to verify precisely what kind of processes are promoting and which learning situations make these processes emerge.

> There have been many processes present in the DU, which is why I consider it to be a DU rich DU in processes and suitable for the 2nd level of ESO. (MD file 21913, translation by the authors)

We can also observe that the percentage of MD that reflect on an indicator when assessing the planning and implementation of the DU is higher than those that use it to justify the proposed improvements.

Figure 3. *Analysis of mathematical processes*

Process and/or Competency	Description	Showed in the DU
Communication	Be able to express learned concepts, ideas, and reasoning.	Yes, from sharing discovery activities and discussions/topic that we did at the beginning of the class to review concepts. We must be more aware of the students who participate and the students who find the topic more difficult.
Exploration	Discover concepts by themselves, exploring the solution possibilities.	Yes, in self-discovery activities.
Formalization	Use the formal language of mathematics.	No. The definitions were created by the students, and while they were corrected and debated, formal language was not used.
Argumentation	Reason and support the statements made.	Yes, it was a key point in most activities.
Problem solving	Solve a non-immediate problem that requires a complex process.	Yes, but not too much. Most of the activities represented small challenges, but they did not become problems. The most intense problems were those of self-knowledge.
Algorithmization	Mechanize a process.	Yes. They were asked to write their own theory, which included the next steps.
Contextualization	Search for the mathematics present in reality.	Yes. Many activities were done in close contexts, such as the institute, hobbies, etc. However not enough importance was given, and it would be good to encourage more the search for mathematical relationships in the reality.
Representation	Use graphs and symbols to express mathematical ideas.	Yes. Especially when working with Cartesian coordinates and function graphs.
Collaborative work	Dialog with colleagues and share ideas to create knowledge.	Yes. Most of the activities were carried out in groups, and much importance was given to collaboration between students.
Modeling	Describe the environment in a mathematical way; model real situations with the mathematics learned.	There was only one exercise in which the students performed the whole modeling process, most of them collected the data and represented it, but they did not come up with any model that would describe it.

Note. MD file 21913 (Authors' translation).

4. Discussion and conclusions

In the initial study of the MD, primary objects, meanings, representations and conversions and situations have emerged from the reflection of teachers in initial training, but, on the other hand, other primary objects, such as propositions, procedures and arguments, also present in the Representativity of Complexity component, have not emerged. Why they have not emerged? Because there is a lack of depth in the reflection of future teachers regarding the propositions, arguments and procedures. The scientific literature contains the following elements that are related to the propositions, procedures, arguments of the notion of function: a) "the procedures consider arbitrariness and univalence as key characteristics of the notion of function"; b) "fundamental statements and procedures related to the notion of function are considered adequate at the educational level" and; c) "situations are promoted in which students must justify their conjectures and procedures" (Pino-Fan & Parra-Urrea, 2021, p. 50). These are not explicit in the MDs we have analyzed.

The fact that only the MD of future teachers is a source of data is one of the limitations of this study. In order to better understand the meta-didactic mathematical knowledge of teachers, it would be necessary to conduct case studies of new teachers and alumni of the master's degree, when they reflect on their own teaching practice using the RIEF. To do this, we would analyze their reflections, make classroom observations, and interview them to learn more about their meta-didactic mathematical knowledge.

In the analysis of the reflections of the MDs on functions for ESO, it can be stated that evidence has been obtained from almost all the RIEF indicators (Table 2), to a greater or lesser extent. However, by going into detail in each MD, it is observed that, when participants review the DU they have designed and its implementation, the guideline— DSCs (Table 1) and a guideline referring to processes (figure 3) — help them reflect (Esqué de los Ojos & Breda, 2021). However, since this is not a specific guideline for the ES of functions (as is the RIEF), the participants do not take into account in their analysis most of the RIEFs. It is found that their reflections have important shortcomings that could influence the quality of their instructional processes.

Failure to consider some of the RIEF indicators may be due to a lack of extended mathematical knowledge of teachers in training as shown by Batista et al. (2022). The use of RIEFs would help to improve this type of knowledge about functions.

Although the literature review indicates that the work that applies DSC as a theoretical-methodological tool has increased in recent years (Malet, 2022), new contexts of use and refinement of the components are needed to analyze teaching processes of specific mathematical topics (Araya et al., 2021; Breda et al., 2021; García Marimón et al., 2021; Piñero-Charlo et al., 2021). Consequently, if teachers were given with tools such as the RIEF, the reflection on their own practice could be improved, since they would have a specific guideline to carry out a more rigorous, clear and efficient analysis. As indicated by Pino-Fan & Parra-Urrea (2021):

> Proper teaching processes about functions require that teachers understand their historical evolution, i.e., that they understand the holistic meaning of the object (its richness of meanings and how to work and promote them) to have a broader and deeper vision of the notion of function. (p. 48)

Therefore, using the RIEF would not only contribute to improve their teaching practice, but would enable a greater meta-didactic-mathematical knowledge of those who use it.

The results of this research show that as educational levels progress, there are new mathematical notions associated with function analysis (slope, continuity, monotony, concavity, etc.), for which it is also necessary to develop a refined tool.

The training courses of future teachers could be enriched with a module in which the RIEF is taught to improve their knowledge of functions by considering all the meanings, representations, processes involved in the complexity of functions and mathematical practices in which these emerge. And they would also delve into the kind of reflection required to design, implement, and reflect on their instructional processes.

Financing

Project PID2021-127104NB-I00 funded by MCIN/ AEI/10.13039/501100011033/ and by FEDER A way to make Europe.

References

Amaya de Armas, T. R., Pino-Fan, L. R. & Medina Rivilla, A. (2016). Evaluación del conocimiento de futuros profesores de matemáticas sobre las transformaciones de las representaciones de una función. *Educación matemática*, *28*(3), 111-144. https://doi.org/10.24844/EM2803.05

Araya, D., Pino-Fan, L., Medrano, I. & Castro, W. F. (2021). Epistemic criteria for the design of tasks about limits on a real variable function. *Bolema*, *35*(69), 179-205. https://doi.org/10.1590/1980-4415v35n69a09

Batista, L. A, Crisóstomo, E. & Macêdo, J. A. (2022). Conocimiento didáctico-matemático movilizado por futuros profesores de matemáticas. *Alteridad*, *17*(2), 194-207. https://doi.org/10.17163/alt.v17n2.2022.03

Ball, D. L., Thames, M. H. & Phelps, G. (2008). Content Knowledge for Teaching What Makes It Special? *Journal of Teacher Education*, 59(5), 389-407. https://doi.org/10.1177/0022487108324554

Braun, V. & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology, 3*(2), 77-101. https://doi.org/10.1191/1478088706qp063oa

Breda, A., Font, V. & Pino-Fan, L. R. (2018). Criterios valorativos y normativos en la didáctica de las matemáticas: El caso del constructo idoneidad didáctica, *Bolema, 32*(60), 255-278.

https://doi.org/10.1590/1980-4415v32n60a13

Breda, A., Pino-Fan, L. R. & Font, V. (2017). Meta Didactic-Mathematical Knowledge of Teachers: Criteria for The Reflection and Assessment on Teaching Practice. *EURASIA Journal of Mathematics, Science and Technology Education*, *13*(6), 1893-1918. https://doi.org/10.12973/eurasia.2017.01207a

Breda, A., Pochulu, M., Sánchez, A. & Font, V. (2021). Simulation of teacher interventions in a training course of mathematics teacher educators. *Mathematics, 9*(24), 3228. https://doi.org/10.3390/math9243228

Elliott, J. (1993). *El cambio educativo desde la investigación-acción*. Ediciones Morata. S.L.

Esqué de los Ojos, D. & Breda, A. (2021). Valoración y rediseño de una unidad sobre proporcionalidad utilizando la herramienta Idoneidad Didáctica. *Uniciencia, 35*(1), 38-54. https://doi.org/10.15359/ru.35-1.3

Flores, M. & Font, V. (2017). Impacto de un programa de desarrollo profesional para profesores de secundaria de matemáticas sobre la enseñanza de funciones en el Ecuador. El caso de la función exponencial. En J. M. Contreras, P. Arteaga, G. R. Cañadas, M. M. Gea, B. Giacomone y M. M. López-Martín (eds.), *Actas del Segundo Congreso Internacional Virtual sobre el Enfoque Ontosemiótico del Conocimiento y la Instrucción Matemáticos* (pp. 1-7). https://bit.ly/46rkxDs

Font, V. (2011). Funciones. Evolución histórica de la noción de función. Algunas ideas clave que nos ofrece la evolución de la noción de función. En J. M. Goñi (coord.), *Matemáticas. Complementos de formación disciplinar* (pp. 145-186). Ed Graó.

Font, V., Calle, E. & Breda, A. (2023). Uso de los criterios de idoneidad didáctica y la metodología Lesson Study en la formación del profesorado de matemáticas en España y Ecuador. *PARADIGMA*, *44*(2), 376-397. https://doi.org/10.37618/PARADIGMA.1011-2251.2023.p376-397.id1424

Font, V., Planas, N. & Godino, J. D. (2010). Modelo para el análisis didáctico en educación matemática. *Infancia y Aprendizaje*, *33*(1), 89-105. https://doi.org/10.1174/021037010790317243

Garcés-Córdova, W. & Font-Moll, V. (2022). Criterios que guían la práctica del profesor de matemáticas en cursos de ciencias básicas para ingeniería. *Uniciencia*, *36*(1), 1-19. https://doi.org/10.15359/ru.36-1.5

García Marimón, O., Diez-Palomar, J., Morales Maure, L. y Durán González, R. E. (2021). Evaluación de secuencias de aprendizaje de matemáticas usando la herramienta de los Criterios de Idoneidad Didáctica. *Bolema, 35*(70), 1047-1072. https://doi.org/10.1590/1980-4415v35n70a23

Godino, J., Batanero, C. y Font, V. (2007). The onto-semiotic approach to research in mathematics education. *ZDM. The International Journal on Mathematics Education*, *39*(1-2), 127-135. https://doi.org/10.1007/s11858-006-0004-1

Godino, J. D., Bencomo, D., Font, V. & Wilhemi, M. R. (2006a). análisis y valoración de la idoneidad didáctica de procesos de estudio de las Matemáticas. *Paradigma*, *27*(2), 221-252. https://bit.ly/46mHiIs

Godino, J. D., Contreras A. & Font, V. (2006b). Análisis de procesos de instrucción basado en el enfoque ontológico-semiótico de la cognición matemática. *Recherches en Didactique des Mathématiques*, *26*(1), 39-88. https://bit.ly/3LDcjzR

Godino, J. D., Giacomone, B., Batanero, C. & Font, V. (2017). Enfoque ontosemiótico de los conocimientos y competencias del profesor de matemáticas. *Bolema*, *31*(57), 90-113. https://doi.org/10.1590/1980-4415v31n57a05

Hart, L. C., Alston, A. S. & Murata, A. (eds.). (2011). *Lesson Study Research and Practice in Mathematics Education*. [Versión de Springer Dordrecht Heidelberg London New York]. https://doi.org/10.1007/978-90-481-9941-9

Leite, L., Verde, A. P., Oliveira, F. & Nunes, J. B. (2021). Mixed methods approach in the theses of a postgraduate program in education: analysis in the light of Creswell. Educação e Pesquisa, 47. https://doi.org/10.1590/S1678-4634202147243789

Malet, O. (2022). *La construcción y aplicación de un dispositivo para la evaluación de idoneidad didáctica de una asignatura masiva del ingreso a la universidad: un recurso para la reflexión profesional*. (Tesis doctoral). Universidad Nacional de Cuyo. Mendoza, Argentina. https://bit.ly/3PS4N6L

Mishra, P. & Koehler, M. J. (2006). Technological Pedagogical Content Knowledge: A new framework for teacher knowledge. *Teachers College Record, 108*(6), 1017-1054. https://doi.org/10.1111/j.1467-9620.2006.00684.x

Morales-López, Y. & Font, V. (2019). Valoración realizada por una profesora de la idoneidad de su clase de matemáticas. *Educação e Pesquisa*, *45,* e189468. https://doi.org/10.1590/S1678-4634201945189468

Morales-Maure, L. (2019). *Competencia de análisis e intervención didáctica del docente de primaria en Panamá.* (Tesis doctoral, Universidad de Barcelona). https://bit.ly/3PS4N6L

Parra-Urrea, Y. E. & Pino-Fan, L. (2017). Análisis ontosemiótico de libros de texto chilenos: el caso del concepto de función. En J. M. Contreras, P., Arteaga, G. R. Cañadas, M. M. Gea, B. Giacomone y M.M. López-Martí (eds.), *Actas del Segundo Congreso Internacional Virtual sobre el Enfoque Ontosemiótico del Conocimiento y la Instrucción Matemáticos*, 1-11. https://bit.ly/461Zeby

Pino-Fan, L. & Parra-Urrea, Y. (2021). Criterios para orientar el diseño y la reflexión de clases sobre funciones. ¿Qué nos dice la literatura científica? *Uno. Revista de didáctica de las matemáticas, 91*, 45-54.

Piñero Charlo, J. C., Ortega García, P. & Román García, S. (2021). Formative potential of the development and assessment of an educational escape room designed to integrate music-mathematical knowledge. *Education Sciences*, *11*, 131. https://doi.org/10.3390/educsci11030131

Ramos, A. B. & Font, V. (2008). Criterios de idoneidad y valoración de cambios en el proceso de instrucción matemática. *Revista Latinoamericana de Investigación en Matemática Educativa, 11*(2), 233-265. https://bit.ly/46mHiIs

Sánchez, A., Breda, A., Font, V. & Sala -Sebastià, G. (2021). ¿Qué errores detectan los futuros profesores en las clases de matemáticas que imparten? *Revista CIDUI, 5*, 1-13.

Schön, D. (1983). *The Reflective Practitioner: How professionals think in action*. Temple Smith.

Seckel, M. J. & Font, V. (2020). Competencia reflexiva en formadores del profesorado de matemáticas. *Magis*, *Revista Internacional de Investigación en Educación*, *12*(25), 127-144. https://doi.org/10.11144/Javeriana.m12-25.crfp

Shulman, L. S. (1987). Knowlendge and Teaching Fundations of the new reform. *Harvard Educational Review*, *57*(1), 1-22. https://bit.ly/3PPkNGL